Many Touchings Force Many Crossings

János Pach and Géza Tóth

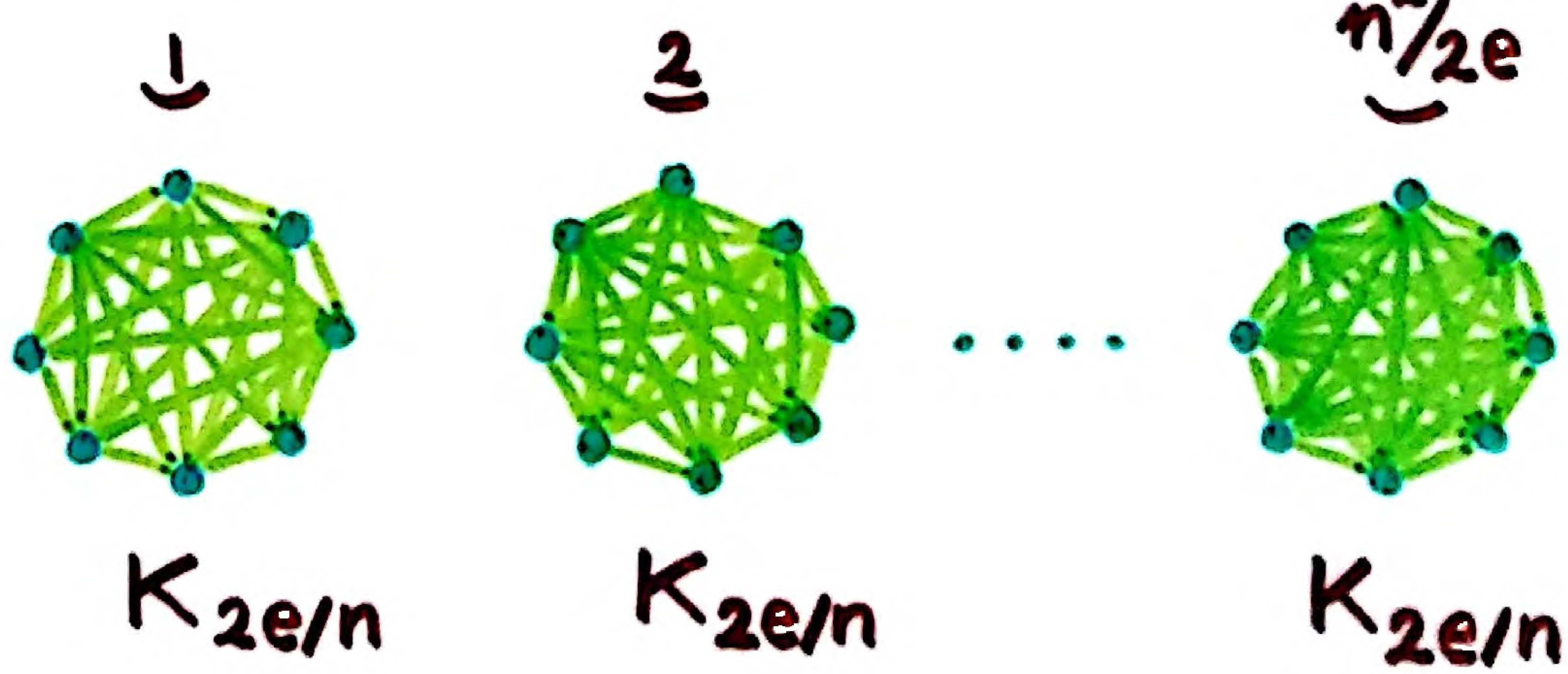
Rényi Institute, Budapest

Theorem (Ajtai-Chvátal-Newborn-Szemerédi 1982, Leighton 1983)

For every simple graph with n vertices and $e \geq 4n$ edges,

$$cr(G) \geq \frac{1}{64} \frac{e^3}{n^2}$$

Construction



T = # touching points 

X = # crossing points 

Theorem (PRT 2017)

For any system of n curves in the plane, no 3 of which pass through the same point, the number of crossing points X satisfies

$$X \geq c\,T\left(\log\log \frac{T}{n}\right)^{1/100},$$

provided that $T \geq c'n$. Here $c, c' > 0$ are constants.

X_{pair} = # CROSSING PAIRS OF CURVES

$\wedge\!\!\backslash$

X = # crossing points

T = # touching points (pairs of curves)

X_{pair} = # CROSSING PAIRS OF CURVES

$\wedge\!\!|$

X = # crossing points

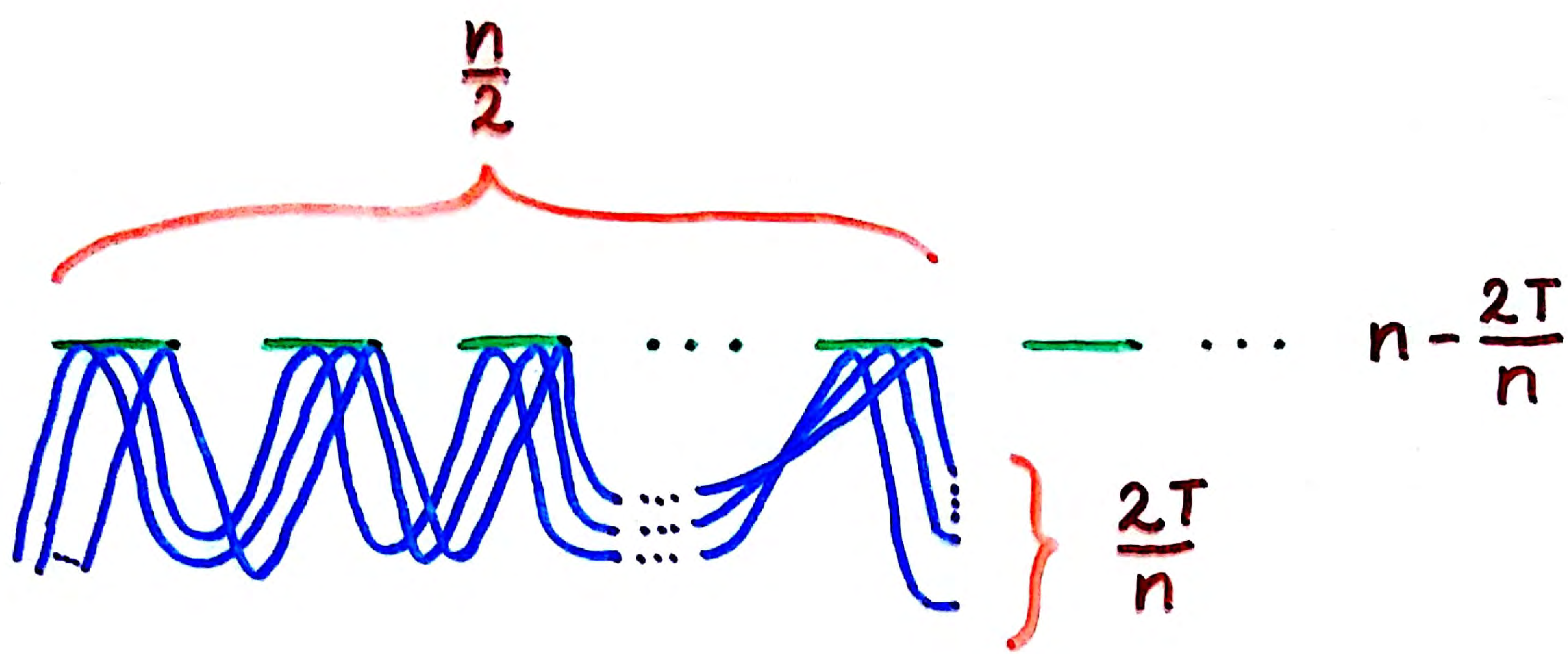
T = # touching points (pairs of curves)

Theorem (PT 2017)

For any family of n curves, no 3 of which pass through the same point, we have

$$X_{pair} \geq \frac{1}{10^5} \frac{T^2}{n^2},$$

provided that $T \geq 10n$.



$$X_{pair} \le \binom{2T/n}{2} \le 2\frac{T^2}{n^2}$$

n curves, <u>touching graph</u> G_t, $|E(G_t)| = T$

<u>crossing graph</u> G_c, $|E(G_c)| = X_{pair}$

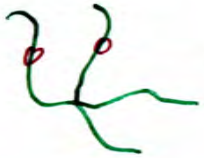
or

We show $\quad X_{pair} \geq \frac{1}{10^5} \cdot \frac{T^2}{n^2} \quad$ if $\quad T > 10n$

case 1.: $\quad T \leq n^{3/2}$

Observation: $\quad X_{pair} = 0 \implies T \leq 3n-6$

Proof: curves $\implies$ points
touchings $\implies$ edges



$\Rightarrow$

PLANAR GRAPH

G_c:

← Minimal edge cover

← Independent

Want: $|E(G_c)| \geq \frac{1}{10^5} \cdot \frac{T^2}{n^2}$

but $\frac{n}{100} \geq \frac{1}{10^5} \cdot \frac{T^2}{n^2}$

← Minimal edge cover

← Has neighbor in V

Isolated

If $|V| \geq \frac{n}{100}$ or $|U'| \geq \frac{n}{100}$: $|E(G_c)| \geq \frac{n}{100}$ ✓

So: $|U_0| \geq \frac{98}{100} n$

If $|E(G_t(V \cup U'))| \geq \frac{t}{10}$: Induction

(many touchings on few curves $\Rightarrow$ many crossings)

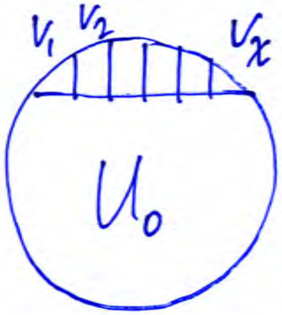
So $|E(G_t(V, U_0))| \geq \frac{t}{2}$ (touchings between V and U_0)

Many touchings between V and U_0.

$x = \chi(G_c(V))$ If x is <u>large</u> $\Rightarrow |E(G_c(V))| \geq \binom{x}{2} \geq \frac{1}{10^5} \cdot \frac{T^2}{n^2}$ ✓

$\left(x \geq \frac{1}{70} \cdot \frac{T}{n}\right)$

So: x is <u>small</u> $\left(x < \frac{1}{70} \cdot \frac{T}{n}\right)$



V_i : color classes, independent.

$V_i \cup U_0$: independent (no crossings) $\Rightarrow$

<u>few touchings</u>

$|E(G_t(V_i, U_0))| \leq 3n \Rightarrow |E(G_t(V, U_0))| \leq x \cdot 3n < \frac{t}{2}$ ↯

contradiction!

Case 2: $T > n^{3/2}$: random sampling