Counterexample to the variant of the Hanani–Tutte Theorem on the Genus 4 Surface

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Results

We disprove a conjecture of Schaefer and Štefankovič [10] from GD 2013 about the extension of the Hanani-Tutte theorem to arbitrary orientable surfaces.

Theorem 1 *There exists a graph G that has a drawing in the compact orientable surfaces $\mathcal{S}$ with 4 handles in which every pair of non-adjacent edges cross an even number of times, but G cannot be embedded in $\mathcal{S}$.*

By taking a disjoint union of G with pairwise disjoint copies of K_5 we obtain a counterexample on an orientable surface of arbitrary genus bigger than 4.
In order to prove the theorem we first give a counterexample to the unified variant (see below) on the torus. Only part 1) is actually needed to prove Theorem 1, but 2) provides a good evidence for why the counterexample works.

Theorem 2 *The following holds.*

1) *The complete bipartite graph $K_{3,4}$ has a drawing $\mathcal{D}$ on the torus with every pair of non-adjacent edges crossing an even number of times, such that for the set W of four vertices in one part every pair of edges with a common endpoint in W crosses an even number of times.*
2) *There is no embedding $\mathcal{E}$ of $K_{3,4}$ on the torus with the same cyclic orders of edges at the vertices of W as in $\mathcal{D}$.*

Introduction

The Hanani–Tutte theorem [5, 11] is a classical result that provides an algebraic characterization of planarity with interesting theoretical and algorithmic consequences, such as a simple polynomial algorithm for planarity testing [9]. The theorem has several variants, the strong and the weak variant are the two most well-known. The notion "the Hanani–Tutte theorem" refers to the strong variant.

The (strong) Hanani–Tutte theorem [5, 11]

A graph is planar if it can be drawn in the plane so that no pair of non-adjacent edges crosses an odd number of times.

The weak Hanani–Tutte theorem [1, 6, 8]

If a graph G has a drawing $\mathcal{D}$ on a compact surface $\mathcal{S}$ where every pair of edges crosses an even number of times, then G has an embedding on $\mathcal{S}$ that preserves the cyclic order of edges at each vertex of $\mathcal{D}$.

Recently a common generalization of both the strong and the weak variant in the plane has been discovered.

Unified Hanani–Tutte theorem [3, 8]

Let G be a graph and let W be a subset of vertices of G. Let $\mathcal{D}$ be a drawing of G where every pair of edges that are independent or have a common endpoint in W cross an even number of times. Then G has a planar embedding where cyclic orders of edges at vertices from W are the same as in $\mathcal{D}$.

The variant of the strong Hanani–Tutte theorem holds for the projective plane. The result was first proved by Pelsmajer, Schaefer and Stasi [7] using the set of minor minimal obstructions to the embeddability of graphs on the projective plane. A direct proof [2] by de Verdière et al. was presented at GD 2016.

The (strong) Hanani–Tutte theorem on the projective plane [2, 7]

If a graph G can be drawn on the projective plane so that no pair of non-adjacent edges crosses an odd number of times, then G can be embedded on the projective plane.

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Proof of Theorem 2

- We give the drawing $\mathcal{D}$ of $K_{3,4}$ on the torus as specified in 1) of Theorem 2.

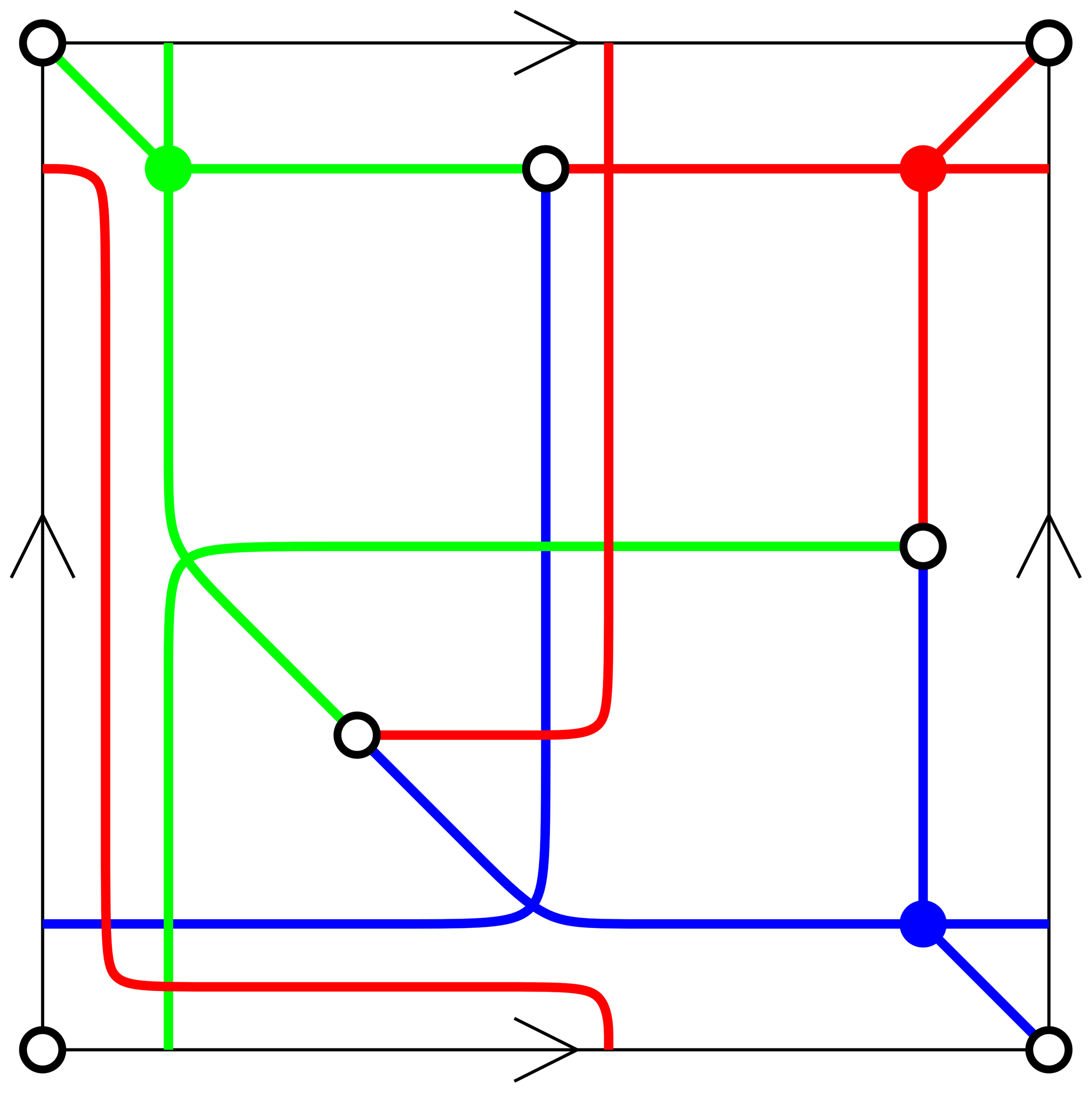

Figure 1: 2-dimensional model of the toroidal drawing $\mathcal{D}$ of $K_{3,4}$ in which every pair of non-adjacent edges cross an even number of times. Vertices in W are drawn as empty circles. The torus is obtained by identifying the opposite sides of the square as indicated by the arrows.

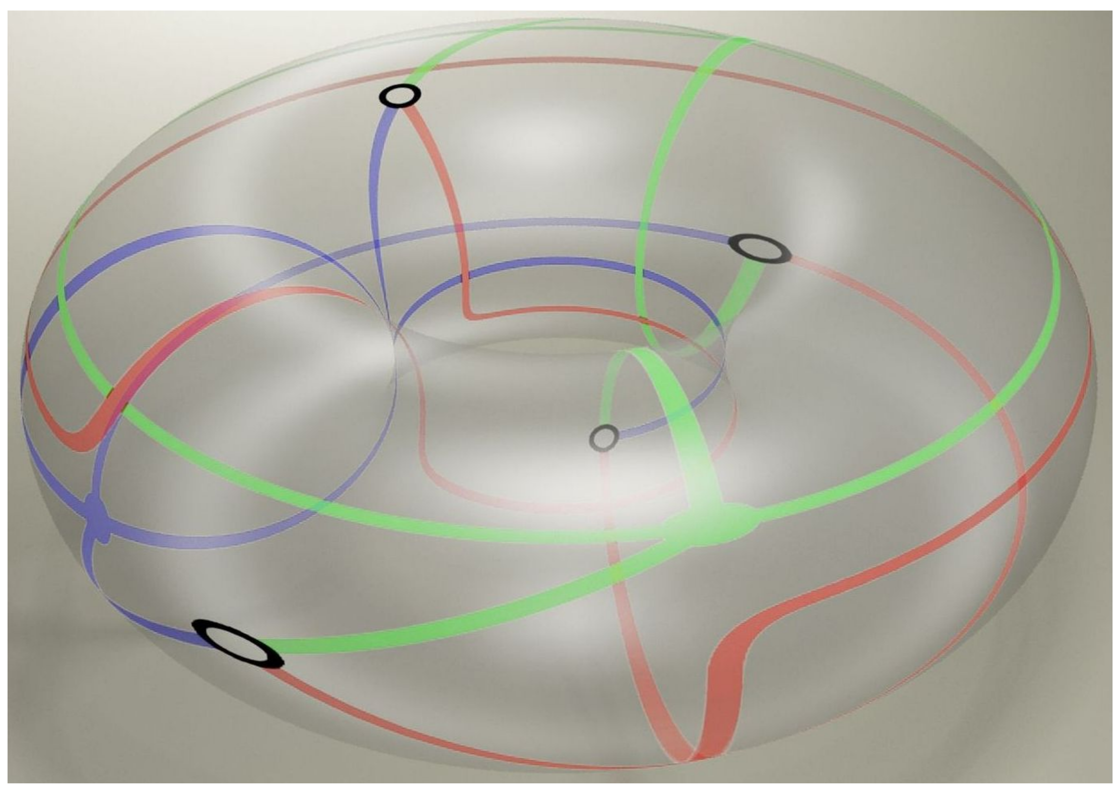

Figure 2: The actual toroidal drawing $\mathcal{D}$ of $K_{3,4}$ from the previous figure realized in the Euclidean 3-space.

- We observe that the counterclockwise cyclic order of the edges around every vertex in W is **red, green** and **blue**, which implies that there are no 4-faces in the embedding $\mathcal{E}$ from 2) of Theorem 2.
- However, no toroidal embedding of $K_{3,4}$ can have all the faces of size at least 6. Indeed, $\mathcal{E}$ has at least 5 faces by Euler's formula $f \geq e - n = 12 - 7 = 5$, and by double-counting the edges we obtain $6f \leq 2e$, which yields $30 = 6 \cdot 5 \leq 2 \cdot 12 = 24$ (contradiction).

Proof of Theorem 1

- The graph G is obtained by combining three disjoint copies of $K_{1,4}$ with a sufficiently large grid by appropriately identifying degree-1 vertices in the three copies of $K_{1,4}$ with vertices in the grid.
- We give a drawing of the graph G on the orientable surface $\mathcal{S}$ of genus 4 in which every pair of non-adjacent edges cross an even number of times. The graph looks like the one in the figure except that the grid is much larger in the actual graph G in comparison with the figure.

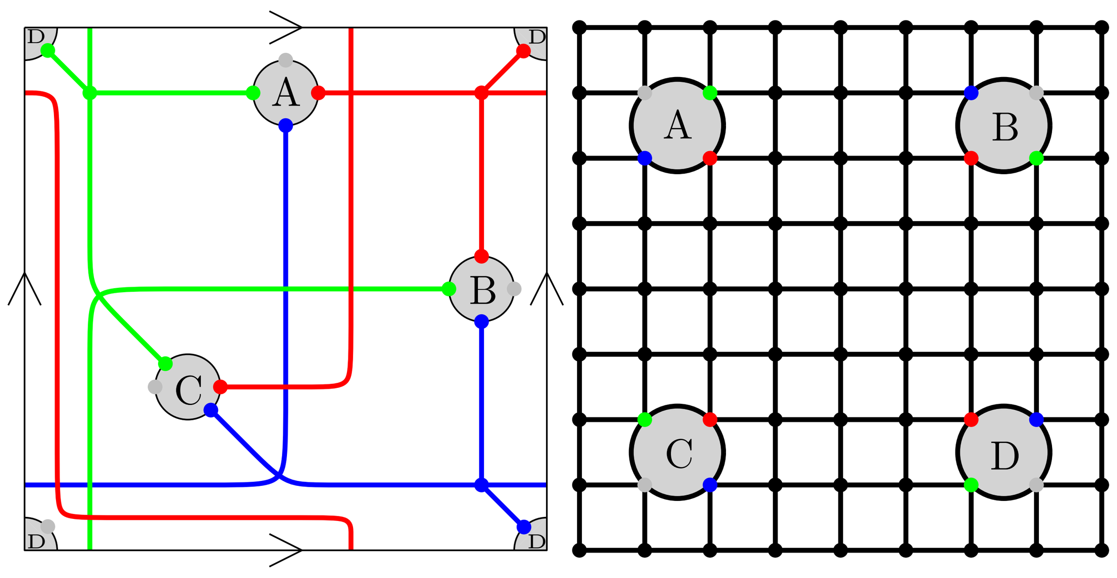

Figure 3: A drawing of G on $\mathcal{S}$, in which every pair of non-adjacent edges cross an even number of times. We drill 4 holes around the vertices of W in $\mathcal{D}$. The drawing is obtained by gluing together along boundaries the obtained torus with 4 holes containing the rest of the drawing $\mathcal{D}$ and an embedding of a large grid on a sphere with 4 holes, where the boundaries of the holes are formed by 4-cycles.

- By Lemma 4 from [4], if the grid in G is sufficiently large we can choose a part of the grid embedded in a planar way and then use the hypothetical embedding of G to embed $K_{4,5}$ as indicated in the figure.

Proof of Theorem 1 (cont')

Figure 4: The surface $\mathcal{S}$ of genus 4 obtained after identifying the boundaries of the 4 holes on the torus with those on the sphere.

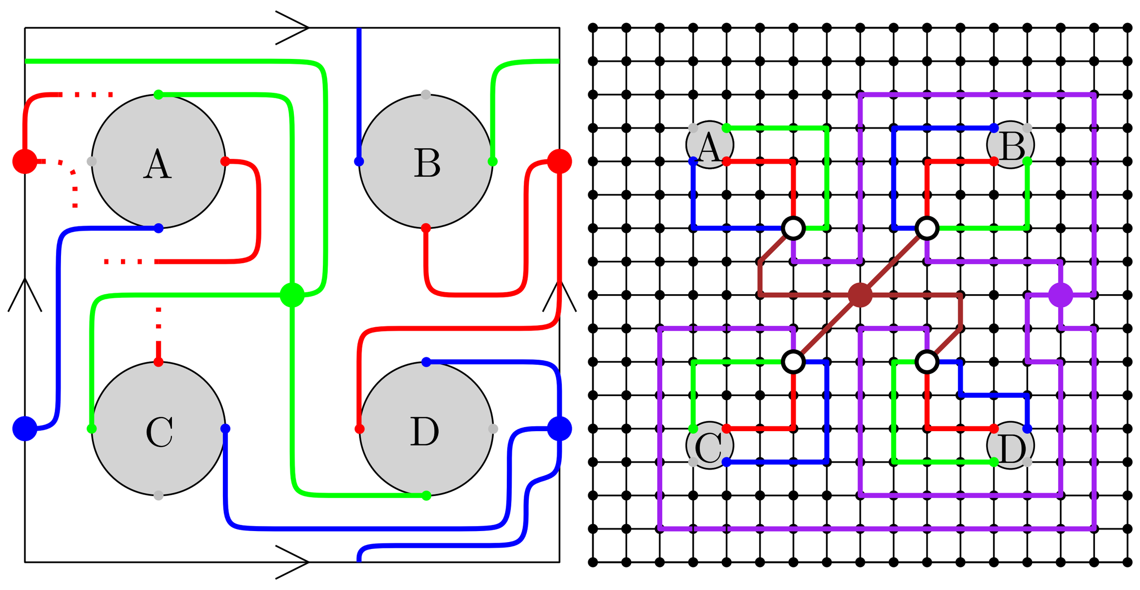

Figure 5: A partial embedding of $K_{4,5}$ on the surface $\mathcal{S}$ of genus 4 drawn by bold polygonal segments. The vertices drawn as empty discs form one part of the vertex set of $K_{4,5}$. The dotted edges cannot be extended without creating an edge crossing.

- We observe that the counterclockwise cyclic order of the edges around every vertex in the smaller part is **brown, purple, green, red** and **blue**, which implies that the faces in such an embedding of $K_{4,5}$ must be at least 10-faces.
- No embedding on $\mathcal{S}$ of $K_{4,5}$ can have all the faces of size at least 10. Indeed, by Euler's formula for the number of faces f we have $f \geq e - n - 6 = 20 - 9 - 6 = 5$, and by double-counting the edges we obtain $10f \leq 2e$, which yields $50 = 10 \cdot 5 \leq 2 \cdot 20 = 40$ (contradiction).

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