On Vertex-Ply and Empty-Ply Proximity Drawings

25th Int. Symp. on Graph Drawing and Network Visualization

September 25-27 2017, Northeastern University, Boston, USA

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Ply of a straight-line drawing
Ply of a straight-line drawing

For each vertex, add a disk centered at it whose radius is half of the longest incident edge.
Ply of a straight-line drawing
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The drawing has ply 3
State of the art

- **Complexity of the testing problem**
  Testing whether a graph admits a drawing with ply 1 is NP-complete (equivalent to recognizing unit-disk contact graphs)
  [Di Giacomo et al. - IISA 2015]
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  Stars and binary trees always admit drawings with ply 2
  
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  Not all bounded-degree trees admit drawings with constant ply
  
  [Angelini et al. - GD 2016]
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  Exponential area is required for stars
  (Max-degree-6) trees: logarithmic ply in polynomial area
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  \[Angelini et al. - GD 2016\] - \[Goodrich.Johnson - Poster today\]

- **Experimental results**
  \[De Luca et al. - WALCOM 2017\]
  \[Heinsohn and Kaufmann - next talk\]
Motivation

• Empirical observation that road networks have low ply, when interpreted as subgraphs of disk intersection graphs

[Eppstein, Goodrich - ACM SIGSPATIAL 2008]
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• Intuition that ply is related to proximity drawings
  [Di Giacomo et al. - IISA 2015]
  Studied in this paper
Proximity drawings

A straight-line drawing in which, for every two vertices $u, v$, their proximity region is empty if (and only if) edge $(u, v)$ exists.

The proximity region of two vertices $u, v \in G$ is a region of the plane determined by their position. Different proximity regions determine different proximity drawings:

- Gabriel graphs
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- Gabriel graphs
- Relative-neighborhood graphs
- Delaunay triangulations
Ply vs. proximity drawings

Proximity region for ply drawings
Ply vs. proximity drawings

Proximity region for ply drawings

Ply depends on disk overlapping and not on vertex-disk containment
Ply vs. proximity drawings

Proximity region for ply drawings

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We define the \textit{vertex-ply} of a drawing as the ply computed only on the points where vertices are placed.
Vertex-ply of a straight-line drawing

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The drawing has vertex-ply 2.
Empty-ply drawings

When vertex-ply $= 1$, we say that the drawing is empty-ply.
Empty-ply drawings

When vertex-ply = 1, we say that the drawing is empty-ply.
What is the ply of an empty-ply drawing?
Relationship between ply and vertex-ply

**Theorem**

An empty-ply drawing has ply at most 5

In general, a drawing with vertex-ply $h$ has ply at most $5h$
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Empty-ply drawings: properties

Property

Empty-ply drawings may be non-planar and non-connected (not true for other proximity drawings)
Empty-ply drawings: properties

**Property**
Empty-ply drawings may be non-planar and non-connected (not true for other proximity drawings)

**Property**
In an empty-ply drawing, the length of adjacent edges differs at most by a factor of 2
Empty-ply drawings: properties

Property

In an empty-ply drawing, no vertex has degree larger than 24
Empty-ply drawings: properties

<table>
<thead>
<tr>
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Empty-ply drawings: properties

Property

In an empty-ply drawing, no vertex has degree larger than 24
The graph $K_{1,24}$ (the star graph with 24 leaves) admits an empty-ply drawing
The graphs $K_{2,12}$, $K_{3,9}$, $K_{4,6}$, $K_7$ admit empty-ply drawings.
Empty-ply drawings of complete graphs

<table>
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The theorem states which complete graphs admit empty-ply drawings and which do not. The graphs $K_{2,12}$, $K_{3,9}$, $K_{4,6}$, $K_7$ admit empty-ply drawings. The graphs $K_{2,15}$ and $K_8$ do not admit empty-ply drawings.
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Empty-ply drawings of trees

Theorem

There exist 4-ary trees (maximum degree 5) not admitting any empty-ply drawing
Empty-ply drawings of trees

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![Diagram of a tree with a root node and several children]
Empty-ply drawings of trees

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**Theorem [Di Giacomo et al. IISA 2015]**
Binary trees (maximum degree 3) admit drawings with ply 2, which are also empty-ply
Ply and vertex-ply of planar drawings

Question [Di Giacomo et al. IISA 2015]

What happens if we restrict to planar drawings?
Is there a trade-off between ply and number of crossings?
Ply and vertex-ply of planar drawings

Observation [Di Giacomo et al. IISA 2015]

The natural drawing of a nested-triangle graph has linear (vertex-)ply, but they admit non-planar drawings with ply 2
Ply and vertex-ply of planar drawings

**Theorem**

Nested-triangle graphs admit planar drawings with ply 4
There exist graphs that require $\Omega(n)$ vertex-ply in any planar drawing, but that admit drawings with ply 5 and 3 crossings.
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Suppose that $(v_1, v_2)$ is the longest outer edge.
Ply and vertex-ply of planar drawings

**Theorem**

There exist graphs that require $\Omega(n)$ vertex-ply in any planar drawing, but that admit drawings with ply 5 and 3 crossings.

Edge $(v_1, v_2)$ is longer than $(u, v_1)$ and $(u, v_2)$. 

[Diagram of a triangle with vertices $v_1$, $v_2$, and $v_3$, and an inner point $u$.]
Triangle \((u, v_1, v_2)\) can be covered with a constant number of circles whose diameter is \(\frac{1}{4}\) of the length of \((v_1, v_2)\)

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One of these circles contains a linear number of $x_1, \ldots, x_m$. 
The disks of these vertices have radius at least $\frac{1}{4}$ of $(v_1, v_2)$.
Thus, each of them contains all the vertices in the circle.

**Theorem**

There exist graphs that require $\Omega(n)$ vertex-ply in any planar drawing, but that admit drawings with ply 5 and 3 crossings.

The disks of these vertices have radius at least $\frac{1}{4}$ of $(v_1, v_2)$.
Thus, each of them contains all the vertices in the circle.
Open problems

- How hard is to test the existence of an empty-ply drawing?
- Do 3-ary trees (max-degree-4) admit empty-ply drawings?
- Do max-degree-3 (planar) graphs admit empty-ply drawings?
- Fill the gaps on complete bipartite graphs. $K_{2,13}$, $K_{3,10}$, $K_{4,7}$, $K_{5,5}$?
- What if we allow at most $k$ vertices to lie inside each disk? Typical generalization for proximity drawings $k$-ply drawings or $k$-empty-ply drawings
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Thanks for your attention!!!