On Upward Drawings of Trees on a Given Grid

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Area Minimization in Planar Straight-Line Drawings

[Krug and Wagner 2008, Biedl 2014]

NP-hard for
• Arbitrary planar graphs
• Planar graphs with bounded pathwidth
• Outerplanar graphs

[M et al. 2008, Biedl 2014]

Polynomial time for
• Planar 3-trees
• Planar graphs with both bounded treewidth and bounded face-degrees
Are Trees Easy?


Minimizing one dimension
• Upward drawings (rooted trees)
• Strictly-upward drawings (rooted trees)
• Non-upward drawings, height (unrooted trees)
Area Minimization for Trees?

NP-hard for
- Ordered trees - under constraints such as isomorphic subtrees must be drawn identically, left and right child must be placed on the left and right of their parent

[NP-hard for
- Drawing ordered/unordered trees on a k-grid, k ∈ {4, 6, 8}, with unit edge length

Area Minimization for Trees?

Straight-line drawings of Trees?

Upward straight-line drawings of rooted Trees?

Strictly upward straight-line drawings of rooted Trees?

Strictly upward straight-line drawings of ordered rooted Trees?
Does $T$ admit a strictly-upward drawing on the given grid?

A strictly-upward drawing of $T$
- Straight-line planar drawing
- Every child is drawn strictly below to its parent
- The ordering of the children can be chosen
Strictly-Upward Drawing on a Given Grid is NP-hard

A reduction from Numerical 3-Dimensional Matching (N3DM)

• **Instance:** Positive integers $r_i, g_i, b_i$, where $1 \leq i \leq k$, and an integer $B$ such that $\sum_i (r_i + b_i + g_i) = k \cdot B$.

• **Question:** Do there exist permutations $\pi$ and $\pi'$ of $\{1, \ldots, k\}$ such that $r_{\pi(i)} + b_i + g_{\pi(i)} = B$ for all $1 \leq i \leq k$?

\[
\begin{array}{ccc}
  r_i & b_i & g_i \\
  5 & 5 & 3 \\
  2 & 7 & 2 \\
  4 & 7 & 7 \\
\end{array}
\]

\[
\sum_i (r_i + b_i + g_i) = k \cdot B \\
42 = 3 \cdot 14
\]

\[
\begin{array}{ccc}
  r_i & b_i & g_i \\
  2 & 5 & 7 \\
  5 & 7 & 2 \\
  4 & 7 & 3 \\
\end{array}
\]

\[
2 + 5 + 7 = 14 \\
5 + 7 + 2 = 14 \\
4 + 7 + 3 = 14
\]
Remains NP-hard under the following restrictions:

- $g_i$s are huge $\in O(k^{4c})$
- $b_i$s are odd and large $\in O(k^{2c})$
- $r_i$s are small $\in O(k^c)$
Strictly-Upward Drawing on a Given Grid is NP-hard

\[ r_i, b_i, g_i \]

\[
\begin{array}{ccc}
5 & 5 & 3 \\
2 & 7 & 2 \\
4 & 7 & 7 \\
\end{array}
\]

Supporting paths
Strictly-Upward Drawing on a Given Grid is NP-hard

$r_i$ $b_i$ $g_i$

5 5 3
2 7 2
4 7 7

Spinal path

Blue stars

$w_{2k+3}$

$u_{2k+3}$

$b_1 = 5$

$b_2 = 7$

$b_3 = 7$
Strictly-Upward Drawing on a Given Grid is NP-hard

\[ r_i \quad b_i \quad g_i \]

\[ 5 \quad 5 \quad 3 \]
\[ 2 \quad 7 \quad 2 \]
\[ 4 \quad 7 \quad 7 \]

Add wall vertices
Strictly-Upward Drawing on a Given Grid is NP-hard

\[ r_i \quad b_i \quad g_i \]

5 5 3
2 7 2
4 7 7

\[ r_1 = 2 \]
\[ r_2 = 5 \]
\[ r_3 = 4 \]

Red stars
Strictly-Upward Drawing on a Given Grid is NP-hard

\[ r_i \quad b_i \quad g_i \]

5 5 3
2 7 2
4 7 7
Strictly-Upward Drawing on a Given Grid is NP-hard

$I$ has an affirmative solution if and only if $T$ admits a drawing on a $(B+4) \times (2k+3)$ grid
From N3DM to Tree Drawing

$r_i \ b_i \ g_i$

\[
\begin{align*}
5 & \ 5 \ 3 \\
2 & \ 7 \ 2 \\
4 & \ 7 \ 7 \\
\end{align*}
\]

\[
\begin{align*}
2 + 5 + 7 & = 14 \\
5 + 7 + 2 & = 14 \\
4 + 7 + 3 & = 14 \\
\end{align*}
\]
From N3DM to Tree Drawing

\[ r_i \ b_i \ g_i \]

\[
\begin{align*}
5 & \quad 5 & \quad 3 \\
2 & \quad 7 & \quad 2 \\
4 & \quad 7 & \quad 7
\end{align*}
\]

\[
\begin{align*}
r_i \; b_i \; g_i & \quad B \\
2 + 5 + 7 & = 14 \\
5 + 7 + 2 & = 14 \\
4 + 7 + 3 & = 14
\end{align*}
\]
From N3DM to Tree Drawing

$r_i \ b_i \ g_i$

5 5 3
2 7 2
4 7 7

$r_i \ b_i \ g_i \ B$

$2 + 5 + 7 = 14$
$5 + 7 + 2 = 14$
$4 + 7 + 3 = 14$

$2k+3$

$B+4$
All the rows (except the topmost row) must be completely used up by the nodes of the tree.
The bottommost layer contains
• two vertices from the supporting paths and
• Wall vertices of the bottommost wall parent.
The next wall parent ($w_3$)
- can have at most two children on $l_8$ and
- all the remaining children lie consecutively on $l_7$

\[
\begin{array}{ccc}
    r_i & b_i & g_i \\
    2 + 5 + 7 & = & 14 \\
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- can have at most two children on \(l_8\) and
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From Tree Drawing to N3DM

The next wall parent ($w_3$)
- can have at most two children on $l_8$ and
- all the remaining children lie consecutively on $l_7$
From Tree Drawing to N3DM

The remaining space is too large for two green stars.
The remaining space is too large for two green stars, and placing two red stars would violate planarity.
The remaining space is too large for two green stars, and placing two red stars would violate planarity.
The remaining space is too large for two green stars, and placing two red stars would violate planarity. Since all grid points must be used up, there can be exactly one green and one red star defining a triple that sum to $B$. 
Future Research

Area Minimization for …

- **Straight-line** drawings of Trees?
- **Upward straight-line** drawings of **rooted** Trees?
- **Strictly upward straight-line** drawings of **rooted** Trees?
- **Strictly upward straight-line** drawings of **ordered rooted** Trees?
<table>
<thead>
<tr>
<th>$r_i$</th>
<th>$b_i$</th>
<th>$g_i$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
<td>14</td>
</tr>
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<td>5</td>
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<td>3</td>
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THANK YOU