The Effect of Planarization on Width

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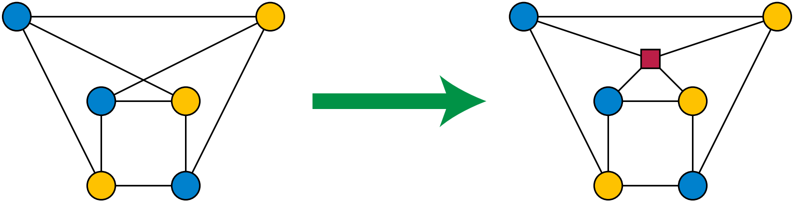
Planarization

Draw a graph with simple crossings (two edges/crossing)

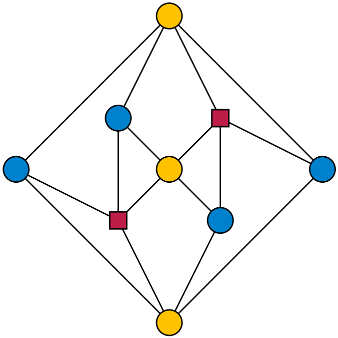
Then, replace each crossing by a new degree-4 vertex

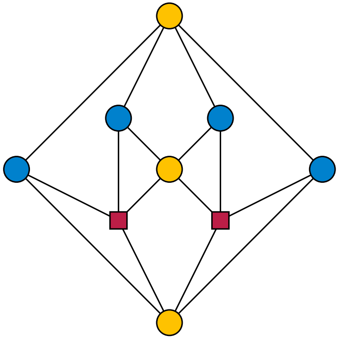


[Garfunkel and Shank 1971; Leighton 1981; Di Battista et al. 2002; Buchheim et al. 2014]

Non-uniqueness

Same graph may have multiple planarizations,
even with minimal # crossings





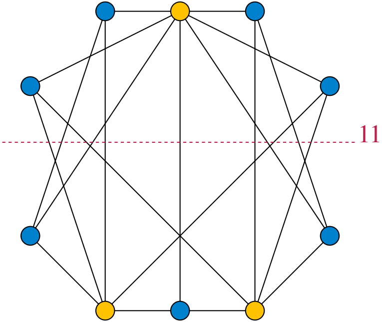
High width ⇒ many crossings

Bisection width: min # edges between subsets of $n/2$ vertices

Small for planar graphs, not decreased by planarization
⇒ high-width graphs have large planarizations
[Leighton 1981]

m-edge graph has $\Omega(m^3/n^2)$ crossings [Leighton 1983]

Unlike probabilistic proof
[Ajtai et al. 1982]
generalizes to nicely-drawn multigraphs [Pach 2017]

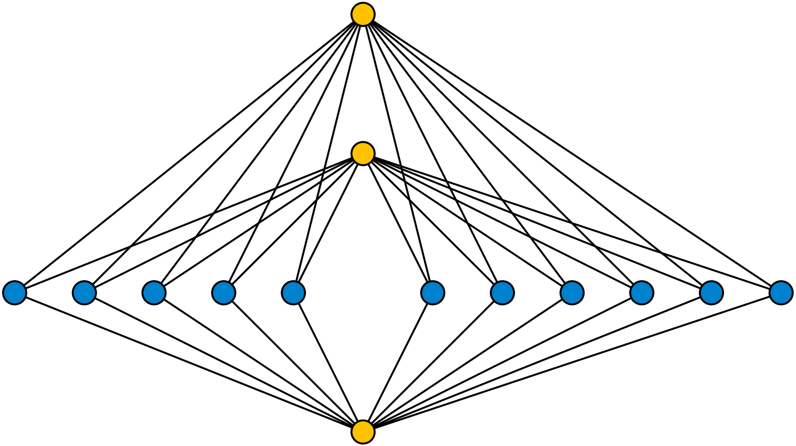


Our question

What can happen when we planarize a low-width graph?

Graphs of treewidth ≤ 2 are already planar

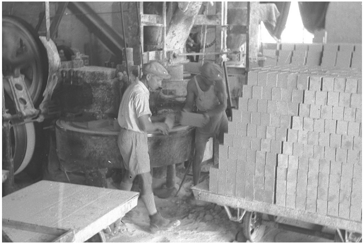
Simplest treewidth-3 nonplanar graphs: $K_{3,n}$



Turán's brick factory problem

The beginning of the study of crossing numbers

Pál Turán was enslaved in a brick factory during World War II



Asked: How to route carts from kilns to storage sites
to minimize # crossings of tracks

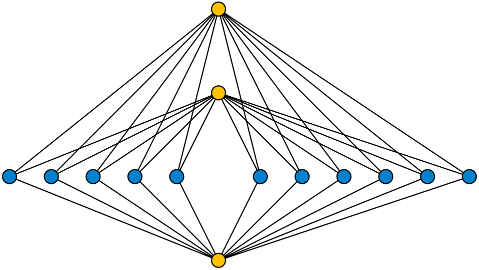
Conjecture: $\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$

Brick factory for $K_{3,n}$

$$\mathrm{cr}(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Proven soon after the war [Zarankiewicz 1954; Urbaník 1955]

Achieved by points on coordinate axes, evenly divided by origin



Also applies to # pairs of crossing edges
(can't reduce # pairs by allowing some pairs to cross many times)

Our main results

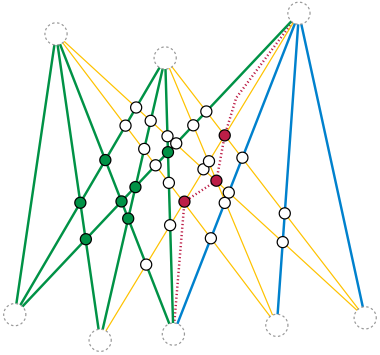
Planarize m curves with c crossing pairs, all crossings simple

$\Rightarrow$ graph has treewidth $\Omega\left(\frac{c}{m}\Big/\log\frac{m^2}{c}\right)$

Proof sketch:

Use separator to partition curves into subsets with a denser intersection graph

Density cannot exceed 1



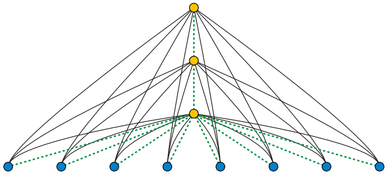
$K_{3,n}$ has $3n$ edges and $\geq n^2/4 - O(n)$ crossing pairs

$\Rightarrow$ Every drawing of $K_{3,n}$ has width $\Omega(n)$

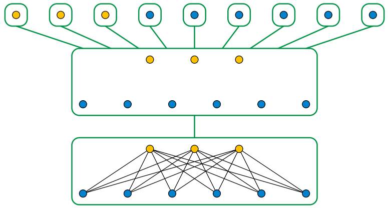
The effect of planarization on width

Corollary: There exist graphs whose width is $O(1)$ but whose planarized width is $\Omega(n)$

Holds for treewidth, pathwidth, branchwidth, tree-depth, and clique-width



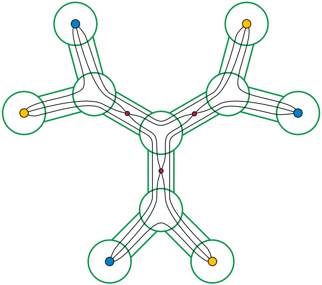
Tree-depth: minimum depth of a tree such that all graph edges are ancestor-descendant pairs



Clique-width: min # colors to construct graph by unions, connecting all pairs with given colors, and recoloring

Some other widths are better behaved

Planarization takes $O(1)$ width $\rightarrow$ $O(1)$ width for
bandwidth, carving width, cutwidth, and
bounded-degree graphs of bounded treewidth or pathwidth



Carving width: Max # graph edges across any edge of a binary tree with graph vertices at leaves

Conclusions

First step in understanding which graph properties are preserved or not preserved by planarization

Many other important properties left for future study

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